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Quantum States of Radiation

Unit II: Density Operator of Radiation, Sudarshan-Glauber Representation, Density Operators of Coherent and Chaotic Radiation, Coherence and Characteristics Functions

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Density Operator Formalism

There are two types of state vectors for different system

(i) PURE STATE

(ii) MIXTURE STATE/STATISTICAL STATE

(i) PURE STATE

If the state of the system can be determined completely by state vector , then state is said to be pure state. The pure state $|\psi\rangle$ can be expanded in a set of eigen kets $\{|u_n\rangle\}$ of an arbitrary physical observable of the system.

$$|\psi\rangle = \sum_n |u_n\rangle\langle u_n|\psi\rangle = \sum_n c_n|u_n\rangle,$$

where $c_n = \langle u_n|\psi\rangle$

$$\langle u_n|u_m\rangle = \delta_{nm} \Longrightarrow \text{Orthonormality condition}$$

$$\sum_n |u_n\rangle\langle u_n| = 1 \Longrightarrow \text{Completeness relation}$$

$$\langle\psi|\psi\rangle = \sum_{n,m} c_n c_m^* \langle u_n|u_m\rangle$$

$$= \sum_{n,m} c_n c_m^* \delta_{nm}$$

$$\langle\psi|\psi\rangle = \sum_n |c_n|^2 = 1$$

Expectation value of any operator F:

$$\langle F\rangle = \langle\psi|F|\psi\rangle$$

$$= \langle\psi|\sum_n |u_n\rangle\langle u_n|F\sum_m |u_m\rangle\langle u_m|\psi\rangle$$

$$= \sum_{n,m}\langle\psi|u_n\rangle\langle u_n|F|u_m\rangle\langle u_m|\psi\rangle$$

$$= \sum_{n,m} c_n^*\langle u_n|F|u_m\rangle c_m$$

$$\langle F\rangle = \sum_{n,m} c_n^* c_m F_{n,m}, \text{ where } F_{n,m} = \langle u_n|F|u_m\rangle$$

(ii) STATISTICAL MIXTURE STATE

It is a generalization of the pure state, where the state of the system is not precisely specified. The existence of the state is defined in terms of probabilities.

e.g. A system in thermal equilibrium i.e. probability of finding states with energy $E_n \propto e^{-\frac{E}{kT}}$.

Probability of finding the system in the state $|\psi_1\rangle$ is P_1

Probability of finding the system in the state $|\psi_2\rangle$ is P_2

………………………………………

Probability of finding the system in the state $|\psi_n\rangle$ is P_n.

Thus, $P_1 + P_2 + P_3 + ... + P_n = \sum_n P_n = 1$.

Hernce, we can say that the system is in the mixture state of $|\psi_1\rangle, |\psi_2\rangle, ...|\psi_n\rangle$ with probabilities $P_1, P_2, ... P_n$.

Density Operator:

The density operator for the pure state is defined as,

$$\rho = |\psi\rangle\langle\psi|$$

$$|\psi\rangle = \sum_n |u_n\rangle\langle u_n|\psi\rangle = \sum_n c_n |u_n\rangle$$

$$\rho = \sum_{n,m} c_n c_m^* |u_n\rangle\langle u_m|$$

then the matrix element of density operator ρ in the basis vectors $|u\rangle$ can be written as,

$$\begin{aligned}
\rho_{kl} &= \langle u_k|\rho|u_l\rangle \\
&= \langle u_k| \sum_{n,m} c_n c_m^* |u_n\rangle\langle u_m|u_l\rangle \\
&= \sum_{n,m} \langle u_k|u_n\rangle\langle u_m|u_l\rangle c_n c_m^* \\
&= \sum_{n,m} \delta_{kn}\delta_{ml} c_n c_m^* \\
\rho_{kl} &= c_k c_l^*
\end{aligned}$$

We may obtain the expectation value of F by means of density operator ρ,

$$\begin{aligned}
\langle F\rangle &= \sum_m \langle u_m|\rho F|u_m\rangle = \sum_{n,m} \langle u_m|\rho|u_n\rangle\langle u_n|F|u_m\rangle \\
&= \sum_{n,m} \rho_{mn} F_{nm} = \sum_{m,n} (\rho F)_{mn} \\
&= Tr(\rho F)
\end{aligned}$$

Normalization condition $\sum_n |c_n|^2 = 1$ can be expressed in terms of density operator as,

$$\sum_n \rho_{nn} = \sum_n c_n c_n^* = \sum_n |c_n|^2 = 1.$$
$$Tr(\rho) = 1$$

Equation of motion for density operator ρ in the Schrödinger picture from its definition:

Schrödinger equation is,

$$i\hbar \frac{\partial |\psi_s(t)\rangle}{\partial t} = H_s |\psi_s(t)\rangle, \qquad (1)$$

Its complex conjugate,

$$-i\hbar \frac{\partial \langle \psi_s(t)|}{\partial t} = \langle \psi_s(t)| H_s \qquad (2)$$

Now

$$\begin{aligned} \frac{d\rho_s}{dt} &= \frac{d}{dt}\left(|\psi_s(t)\rangle\langle\psi_s(t)|\right) \\ &= \frac{d}{dt}\left(|\psi_s(t)\rangle\right)\langle\psi_s(t)| + |\psi_s(t)\rangle \frac{d}{dt}\left(\langle\psi_s(t)|\right) \end{aligned} \qquad (3)$$

Substitute the value of eq.(1) and eq.(2) in eq.(3), we get

$$\begin{aligned} \frac{d\rho_s}{dt} &= \left(\frac{1}{i\hbar} H_s |\psi_s(t)\rangle\right)\langle\psi_s(t)| + |\psi_s(t)\rangle\left(-\frac{1}{i\hbar}\langle\psi_s(t)| H_s\right) \\ &= \frac{1}{i\hbar}(H_s\rho_s - \rho_s H_s) = \frac{1}{i\hbar}[H_s, \rho_s] \end{aligned} \qquad (4)$$

where $\rho_s = |\psi_s(t)\rangle\langle\psi_s(t)|$

Similarly equation of motions for density operator ρ in Heisenberg and Interaction picture are

$$\frac{d\rho_H}{dt} = \frac{1}{i\hbar}[H_H, \rho_H]; \rho_H = |\psi_H(t)\rangle\langle\psi_H(t)|$$

$$\frac{d\rho_I}{dt} = \frac{1}{i\hbar}[H_I, \rho_I]; \rho_I = |\psi_I(t)\rangle\langle\psi_I(t)|$$

For pure state:

$$\text{Density operator is hermitian i.e. } \rho^{+} = \rho$$

$$\text{and also } \rho^2 = \rho, \; Tr(\rho^2) = 1$$

For impure state:

$$\rho_{kl} = \overline{c_k c_l^*} \text{ (average ensemble)}$$

$$Tr(\rho) = \sum_k \rho_{kk} = \sum_k \overline{|c_k|^2} = 1$$

$$\langle F \rangle = \sum_{n,m} \overline{c_n c_m^*} F_{n,m}$$

$$\langle F \rangle = \sum_{n,m} \rho_{n,m} F_{n,m} = Tr(\rho F)$$

(i) If the system is in pure state

$$\rho = |\psi\rangle\langle\psi|, \; \rho^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho$$

(ii) If the system is not in pure state, $\rho^2 \neq \rho$

For a system described by a statistical mixture state, the density operator in the schrodinger picture can be defined by,

$$\rho_s = \sum_k p_k \left|\psi_k^s(t)\right\rangle\left\langle\psi_k^s(t)\right|$$

$$\rho_s = \sum_k p_k \rho_k^s, \text{ where } \rho_k^s = \left|\psi_k^s(t)\right\rangle\left\langle\psi_k^s(t)\right|$$

The density operators in Heisenberg picture and in the interaction picture are

$$\rho_H = \sum_k p_k \rho_k^H$$

$$\rho_I = \sum_k p_k \rho_k^I$$

$$\text{So, } \langle A \rangle = Tr(\rho_s A) = Tr(\rho_H A) = Tr(\rho_I A)$$

Thus,

$$\rho^2 = \sum_k p_k|\psi_k\rangle\langle\psi_k|\sum_{k'} p_{k'}|\psi_{k'}\rangle\langle\psi_{k'}|$$
$$= \sum_{k,k'} p_k p_{k'}|\psi_k\rangle\langle\psi_k|\psi_{k'}\rangle\langle\psi_{k'}|$$
$$= \sum_{k,k'} p_k p_{k'}|\psi_k\rangle\delta_{kk'}\langle\psi_{k'}|$$
$$= \sum_k p_k^2|\psi_k\rangle\langle\psi_k|$$
$$\neq \rho$$

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}\left(\sum_k p_k^2|\psi_k\rangle\langle\psi_k|\right)$$
$$= \langle\psi_k|p_k^2|\psi_k\rangle$$
$$= p_k^2\langle\psi_k|\psi_k\rangle$$
$$= p_k^2$$

Since $0 < p_k < 1$

$\mathrm{Tr}(\rho^2) = p_k^2$ hence $\mathrm{Tr}(\rho^2) \le 1$, equality holds for pure state.

Diagonal or P representation (Sudarshan-Glauber representation):

If sysrme is in a pure state $|\psi\rangle$ then density operator $\rho = |\psi\rangle\langle\psi|$,

Density operator for radiation as

$$\rho = \int d^2z\, \phi(z)|z\rangle\langle z|,$$

where $\phi(z)$ = Weight function can be thought similar to probability distribution function.

$\phi(z)d^2z$ = interpreted as probability for classical amplitude to take value between z and $z + dz$ .i.e. z_r to $z_r + dz_r$, z_i to $z_i + dz_i$

$$d^2z = dz_r dz_i$$

$\langle F\rangle = \langle\psi|F|\psi\rangle$

$$\text{i.e } \langle F\rangle = \mathrm{Tr}(\rho F)$$

$$\langle a^{+m}a^n\rangle = Tr(\rho a^{+m}a^n)$$
$$= \sum_k \langle k|\rho a^{+m}a^n|k\rangle$$
$$= \sum_k \langle k|\int d^2z\, \phi(z)|z\rangle\langle z|a^{+m}a^n|k\rangle$$
$$= \sum_k \int d^2z\, \langle k|z\rangle\, \phi(z)\langle z|a^{+m}a^n|k\rangle$$
$$= \sum_k \int d^2z\, \phi(z)\langle z|a^{+m}a^n|k\rangle\langle k|z\rangle$$
$$= \int d^2z\, \phi(z) z^{*m} z^n$$

Glauber Sudarshan representation: $\rho = \int d^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|$ for single mode radiation.

$P(\alpha) \geq 0$, always positive definite; $P(\alpha) \equiv$ Weight function; $\rho \equiv$ density operator

since ρ is hermitian, $\rho^+ = \rho$; $(P(\alpha))^* = P(\alpha) \Rightarrow P(\alpha)$ is real. $d^2\alpha\, P(\alpha)$ associate with probability distribution function. It has values lying between α and $\alpha + d^2\alpha$. In diagonal P-representation, weight function $P(\alpha)$ always has to be positive definite.

P-representation for coherent state:

Coherent state $|\alpha_0\rangle$; density operator: $\rho = |\alpha_0\rangle\langle\alpha_0|$

P- representation $\rho = \int d^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha| = \int d^2\alpha\, \delta^2(\alpha - \alpha_0)|\alpha\rangle\langle\alpha|$, where

$P(\alpha) = \delta^2(\alpha - \alpha_0)$

$\delta^2(\alpha - \alpha_0) = \delta(\alpha_r - \alpha_{0r})\delta(\alpha_i - \alpha_{0i})$

Normally Ordered and Antinormally Ordered Functions:

In normal ordering all creation operators appears on the left and annihilation operators on the right, while in antinormal ordering all creation operators on the right and annihilation operators appears on the left.

e.g. a^+a is normal ordering, aa^+ is antinormal ordering.

Problem: Convert Normal Order function $a^{+2}a^2$ into antinormal function.

Solution: $a^{+2}a^2 = a^+a^+aa$

We know that $[a,\ a^+] = 1$ so $a^+a = aa^+ - 1$

Therefore

$$a^{+2}a^2 = a^+a^+aa = a^+(aa^+ - 1)a = a^+aa^+a - a^+a$$

$$
\begin{aligned}
a^{+2}a^{2} &= (aa^{+}-1)(aa^{+}-1)-(aa^{+}-1) \\
&= aa^{+}aa^{+}-aa^{+}-aa^{+}+1-aa^{+}+1 \\
&= a(aa^{+}-1)a^{+}-3a^{+}a+2 \\
&= aaa^{+}a^{+}-4a^{+}a+2 \\
&= a^{2}a^{+2}-4a^{+}a+2
\end{aligned}
$$

Normally Ordered Coherence function:

$$\Gamma^{(n,m)} = Tr(\rho a^{+m}a^{n})$$

which is coherence function of order n, m

$$
\begin{aligned}
\Gamma^{(n,m)} &= Tr(\rho a^{+m}a^{n}) \\
&= \sum_{k} \langle k | \int d^{2}\alpha\, P(\alpha) |\alpha\rangle\langle\alpha| a^{+m}a^{n}) | k\rangle \\
&= \int d^{2}\alpha\, P(\alpha) \sum_{k} \langle k|\alpha\rangle\langle\alpha| a^{+m}a^{n} |k\rangle \\
&= \int d^{2}\alpha\, P(\alpha) \langle\alpha| a^{+m}a^{n} |\alpha\rangle \sum_{k} |k\rangle\langle k| \\
&= \int d^{2}\alpha\, P(\alpha) \langle\alpha| a^{+m}a^{n} |\alpha\rangle
\end{aligned}
$$

$$\Gamma^{(n,m)} = \int d^{2}\alpha\, P(\alpha)\, \alpha^{*m}\, \alpha^{n}$$

Thus Coherence functions are moments of the weight function $P(\alpha)$

Density Operator for Single mode radiation in thermal equilibrium at temperature, T:

Consider a system exists in thermal equilibrium with reservoir at temperature T. From statistical mechanics, probability of finding system in state E_n is directly proportional to Boltzmann factor, i.e.

$$E_n \propto e^{-E_n/kT}$$

$$p_n = \frac{e^{-E_n/kT}}{Z},$$

where $Z = \sum_m e^{-E_m/kT}$ is called partition function

Therefore
$$p_n = \frac{e^{-E_n/kT}}{\sum_m e^{-E_m/kT}} \quad (1)$$

The same result of equation (1) can be expressed quantum mechanically by taking density operator ρ as

$$p_n = \frac{e^{-E_n/kT}}{Tr(\rho e^{-H/kT})} \quad (2)$$

The probability for existence in the nth state is

$$\langle n|\rho|n\rangle = \frac{e^{-E_n/kT}}{\sum_m e^{-E_m/kT}} \quad (3)$$

Note: Show that equation (1) and equation(3) are same.

$$H|n\rangle = E_n|n\rangle$$

$$e^{-H/kT}|n\rangle = e^{-E_n/kT}|n\rangle$$

$$\langle n|e^{-H/kT}|n\rangle = e^{-E_n/kT}$$

$$\therefore Tr(e^{-H/kT}) = \sum_n \langle n|e^{-H/kT}|n\rangle = \sum_n e^{-E_n/kT}$$

Hence, $$\langle n|\rho|n\rangle = \frac{e^{-E_n/kT}}{\sum_m e^{-E_m/kT}}$$

$$e^{-H/kT} = e^{-E_n/kT}\sum_n |n\rangle\langle n|$$

Since $H = (N+\frac{1}{2})\omega$

$$e^{-H/kT} = e^{-(N+\frac{1}{2})\omega/kT}\sum_n |n\rangle\langle n|$$

$$= \sum_n e^{-(n+\frac{1}{2})\omega/kT}|n\rangle\langle n| \qquad \text{since } N|n\rangle = n|n\rangle$$

$$Tr(e^{-H/kT}) = \sum_n \langle n|e^{-H/kT}|n\rangle$$

$$= \sum_n e^{-(n+\frac{1}{2})\omega/kT}$$

$$= e^{-\omega/2kT}[1 + e^{-\omega/kT} + e^{-2\omega/kT} + e^{-3\omega/kT} + ...$$

$$Tr(e^{-H/kT}) = e^{-\omega/2kT}\left(\frac{1}{1-e^{-\omega/kT}}\right)$$

Density operator,

$$\rho = \frac{e^{-H/kT}}{Tr(e^{-H/kT})}$$

$$= \frac{\sum_n e^{-(n+\frac{1}{2})\omega/kT}|n\rangle\langle n|}{e^{-\omega/2kT}\left(\frac{1}{1-e^{-\omega/kT}}\right)}$$

$$= \frac{e^{-\omega/2kT}\sum_n e^{-n\omega/kT}|n\rangle\langle n|}{e^{-\omega/2kT}\left(\frac{1}{1-e^{-\omega/kT}}\right)}$$

$$= \frac{\sum_n e^{-n\omega/kT}|n\rangle\langle n|}{\left(\frac{1}{1-e^{-\omega/kT}}\right)}$$

$$\rho = \sum_n e^{-n\omega/kT}\left(1-e^{-\omega/kT}\right)|n\rangle\langle n|$$

which is density operator for thermal radiation (chaotic light)

The average number of photons in this mode

$$\bar{n} = Te[N\rho] = \sum_{n=0}^{\infty}\langle m|N\rho|m\rangle$$

$$= \sum_{n=0}^{\infty}\langle m|N\sum_n e^{-n\omega/kT}\left(1-e^{-\omega/kT}\right)|n\rangle\langle n|m\rangle$$

$$= \sum_{n=1}^{\infty} ne^{-n\omega/kT}\left(1-e^{-\omega/kT}\right)$$

$$= \left(1-e^{-\omega/kT}\right)\sum_{n=1}^{\infty} ne^{-n\omega/kT}$$

$$= \left(1-e^{-\omega/kT}\right)\left(e^{-\omega/kT} + 2e^{-2\omega/kT} + 3e^{-3\omega/kT} + \ldots\ldots\right)$$

$$= \left(1-e^{-\omega/kT}\right)e^{-\omega/kT}\left(1 + 2e^{-\omega/kT} + 3e^{-\omega/kT} + \ldots\ldots\right)$$

Put $e^{-\omega/kT} = x$

$$\bar{n} = Te[N\rho] = (1-x)\,x\left(1+2x+3x^2+\ldots\ldots\right)$$

$$= (1-x)\,x\frac{d}{dx}\left(1+x+x^2+3x^3+\ldots\ldots\right)$$

$$= (1-x)\,x\frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= (1-x)\,x\frac{1}{(1-x)^2}$$

$$= \frac{x}{(1-x)} = \frac{e^{-\omega/kT}}{\left(1-e^{-\omega/kT}\right)}$$

$$e^{-\omega/kT} = \bar{n}\left(1-e^{-\omega/kT}\right)$$

$$= \bar{n} - \bar{n}e^{-\omega/kT}$$

$$\bar{n} = \bar{n}e^{-\omega/kT} + e^{-\omega/kT} = e^{-\omega/kT}(\bar{n}+1)$$

$$e^{-\omega/kT} = \frac{\bar{n}}{(\bar{n}+1)}$$

$$1-e^{-\omega/kT} = 1-\frac{\bar{n}}{(\bar{n}+1)} = \frac{1}{(\bar{n}+1)}$$

Hence,

$$\rho = \sum_n e^{-n\omega/kT}\left(1-e^{-\omega/kT}\right)|n\rangle\langle n|$$

$$= \sum_n \left(\frac{\bar{n}}{\bar{n}+1}\right)^n \left(\frac{1}{\bar{n}+1}\right)|n\rangle\langle n|$$

$$= \sum_n \left(\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}\right)|n\rangle\langle n|$$

which is density operator for single mode radiation at thermal equilibrium at temperature T.

$$Tr(\rho) = \langle n|\rho|n\rangle$$

$$\rho = \sum_n \left(\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}\right)|n\rangle\langle n|$$

$$\therefore Tr(\rho) = \langle n|\sum_n \left(\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}\right)|n\rangle$$

$$= \sum_n \left(\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}\right)$$

$$= \frac{1}{(\bar{n}+1)}\sum_n \left(\frac{\bar{n}}{(\bar{n}+1)}\right)^n$$

$$= \frac{1}{(\bar{n}+1)}\left(\frac{1}{1-\dfrac{\bar{n}}{(\bar{n}+1)}}\right) = 1$$

Hence $$Tr(\rho) = 1$$

Mehta – Sudarshan Method:

P- representation or Glauber- Sudarshan representation is

$$\rho = \int d^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|$$

Evaluate: $\langle -\beta|\rho|\beta\rangle = \langle -\beta|\int d^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|\beta\rangle = \int d^2\alpha\, P(\alpha)\langle -\beta|\alpha\rangle\langle\alpha|\beta\rangle$

Where, $\langle -\beta|\alpha\rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 - \beta^*\alpha\right)\right]$

$$\langle \alpha|\beta\rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 + \alpha^*\beta\right)\right]$$

$$\langle -\beta|\rho|\beta\rangle = \int d^2\alpha\, P(\alpha) \exp\left[-\left(|\alpha|^2 + |\beta|^2 - \beta^*\alpha + \alpha^*\beta\right)\right]$$

$$\langle -\beta|\rho|\beta\rangle e^{|\beta|^2} = \int d^2\alpha\, P(\alpha) \exp\left[-\left(|\alpha|^2 - \beta^*\alpha + \alpha^*\beta\right)\right]$$

Therefore

$$P(\alpha)e^{-|\alpha|^2} = \frac{1}{\pi^2}\int d^2\beta \langle -\beta|\rho|\beta\rangle \exp\left[\left(-\alpha^*\beta + \beta^*\alpha\right)\right] e^{|\beta|^2}$$

Use Fourier transform,

$$g(\beta) = \int d^2\alpha\, f(\alpha) \exp\left[\left(\alpha^*\beta - \beta^*\alpha\right)\right]$$

$$f(\alpha) = \frac{1}{\pi^2}\int d^2\beta\, g(\beta) \exp\left[-\left(\alpha^*\beta - \beta^*\alpha\right)\right]$$

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\beta \langle -\beta|\rho|\beta\rangle \exp\left[\left(-\alpha^*\beta + \beta^*\alpha\right)\right] e^{|\alpha|^2 + |\beta|^2}$$

$P(\alpha)$ for thermal radiation:

$$\rho = \sum_n \left(\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}\right)|n\rangle\langle n|$$

$$\langle -\beta|\rho|\beta\rangle = \sum_n \left(\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}\right)\langle -\beta|n\rangle\langle n|\beta\rangle$$

$$|\beta\rangle = e^{-\frac{1}{2}|\beta|^2} \sum_m \frac{\beta^m}{\sqrt{m!}}|n\rangle$$

$$\langle -\beta|\rho|\beta\rangle = \sum_n \left(\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}\right)\langle -\beta|n\rangle\langle n|\beta\rangle$$

$$= \sum_n \left(\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}\right) e^{-\frac{1}{2}|\beta|^2} \sum_n \frac{\beta^n}{\sqrt{n!}}\, e^{-\frac{1}{2}|\beta|^2} \sum_n \frac{-\beta^{*n}}{\sqrt{n!}}$$

$$= \frac{e^{-|\beta|^2}}{(\bar{n}+1)} \sum_n \left(\frac{\bar{n}}{\bar{n}+1}\right)^n \sum_n \frac{-|\beta|^{2n}}{n!}$$

$$= e^{-|\beta|^2}(\bar{n}+1)^{-1}\exp\left(-\frac{\bar{n}|\beta|^2}{\bar{n}+1}\right)$$

Therefore,

$$P(\alpha) = \frac{1}{\pi^2}\int d^2\beta \langle -\beta|\rho|\beta\rangle \exp[(\alpha\beta^* - \beta\alpha^*)] e^{|\alpha|^2+|\beta|^2}$$

$$= \frac{1}{\pi^2} e^{-|\beta|^2}(\bar{n}+1)^{-1}\int d^2\beta \exp\left(-\frac{\bar{n}|\beta|^2}{\bar{n}+1}\right) \exp[(\alpha\beta^* - \beta\alpha^*)] e^{|\alpha|^2+|\beta|^2}$$

$$= \frac{1}{\pi^2} e^{|\alpha|^2}(\bar{n}+1)^{-1}\int d^2\beta \exp\left(-\frac{\bar{n}|\beta|^2}{\bar{n}+1} + \alpha\beta^* - \beta\alpha^*\right)$$

$$\int d^2\beta \exp\left(-A|\beta|^2 + B\beta + C\beta^*\right) = \frac{\pi}{A}\exp\left(\frac{BC}{A}\right);$$

Where $A = \frac{\bar{n}}{\bar{n}+1}$, $B = -\alpha^*$, $C = \alpha$,

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Evaluation of Integration

$$I = \int d^2\beta \exp\left(-A|\beta|^2 + B\beta + C\beta^*\right) = \int d^2\beta \exp\left(-A|\beta|^2\right) \sum_{m,n} \frac{B^n\beta^n C^m \beta^{*m}}{m!n!}$$

Write $\beta = Xe^{i\theta}$; $d^2\beta = XdXd\theta$

$$I = \int_0^\infty dX\ Xe^{-AX^2} \int_0^{2\pi} d\theta \sum_{m,n} \frac{B^n C^m X^{n+m} e^{i(n-m)\theta}}{m!n!}$$

Integration on θ gives zero if $n \neq m$,

$$\int_0^{2\pi} d\theta e^{i(n-m)\theta} = 2\pi\delta_{nm}$$

$$I = 2\pi\delta_{nm}\int_0^{\infty} dX \;\; X e^{-AX^2} \sum_{m,n} \frac{B^n C^m X^{n+m}}{m!n!} ..$$

$$= 2\pi\sum_n \frac{B^n C^n}{(n!)^2}\int_0^{\infty} dX \;\; X^{2n+1} e^{-AX^2}$$

If $u = X^2 ; du = 2XdX \;\; ; \;\; \frac{du}{2} = XdX$

$$\int_0^{\infty} dX \;\; X^{2n+1} e^{-AX^2} = \frac{n!}{A^{n+1}}$$

$$I = \pi\sum_n \frac{(BC)^n}{(n!)^2}\frac{n!}{A^{n+1}} = \frac{\pi}{A}\exp\left(\frac{BC}{A}\right)$$

$$\int d^2\beta \exp\left(-A|\beta|^2 + B\beta + C\beta^*\right) = \frac{\pi}{\bar{n}}(\bar{n}+1)\exp\left(-\frac{(\bar{n}+1)|\alpha|^2}{\bar{n}}\right)$$

hence,

$$P(\alpha) = \frac{1}{\pi^2(\bar{n}+1)} e^{|\alpha|^2} \frac{\pi}{\bar{n}}(\bar{n}+1)\exp\left(-\frac{(\bar{n}+1)|\alpha|^2}{\bar{n}}\right)$$

$$= \frac{1}{\pi\bar{n}}\exp\left(-|\alpha|^2\left(\frac{\bar{n}+1}{\bar{n}} - 1\right)\right)$$

$$= \frac{1}{\pi\bar{n}}\exp\left(-\frac{|\alpha|^2}{\bar{n}}\right)$$

which is Gaussian distribution. where, average number of photons $\bar{n} = \frac{e^{-\omega/kT}}{\left(1 - e^{-\omega/kT}\right)}$

Q- Representation:

For evaluating the antinormally ordered coherence function,

$$\Gamma_A^{(n,m)} = Tr[\rho a^n a^{+m}]$$

$$\begin{aligned}
Tr[\rho a^n a^{+m}] &= \sum_n \langle n|\rho a^n a^{+m}|n\rangle \\
&= \sum_n \langle n|\rho a^n \frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| a^{+m}|n\rangle \\
&= \pi^{-1}\int d^2\alpha\, \langle\alpha|a^{+m}\rho a^n|\alpha\rangle \\
&= \pi^{-1}\int d^2\alpha\, \langle\alpha|\rho|\alpha\rangle \alpha^{*m}\alpha^n
\end{aligned}$$

Thus moments of $\langle\alpha|\rho|\alpha\rangle$ give the antinormally ordered coherence functions.

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$$

$$\rho = \sum_\psi P_\psi |\psi\rangle\langle\psi|;\ P_\psi \ge 0$$

$$\begin{aligned}
Q(\alpha) &= \frac{1}{\pi}\langle\alpha|\sum_\psi P_\psi|\psi\rangle\langle\psi|\alpha\rangle \\
&= \sum_\psi \frac{1}{\pi} P_\psi \langle\alpha|\psi\rangle\langle\psi|\alpha\rangle \\
&= \sum_\psi \frac{1}{\pi} P_\psi |\langle\psi|\alpha\rangle|^2
\end{aligned}$$

$$\therefore Q(\alpha) \ge 0$$

$$\begin{aligned}
\int d^2\alpha Q(\alpha) &= \frac{1}{\pi}\int d^2\alpha \langle\alpha|\rho|\alpha\rangle \\
&= \frac{1}{\pi}\int d^2\alpha \langle\alpha|\sum_n |n\rangle\langle n|\rho|\alpha\rangle \\
&= \sum_n \langle n|\rho \frac{1}{\pi}\int d^2\alpha |\alpha\rangle\langle\alpha|n\rangle \\
&= \sum_n \langle n|\rho|n\rangle = Tr(\rho) = 1
\end{aligned}$$

R- Representation:

Completeness relation for coherent state is

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = 1$$

$$\rho = \pi^{-1}\int d^2\alpha|\alpha\rangle\langle\alpha|\rho\,\pi^{-1}\int d^2\beta|\beta\rangle\langle\beta|$$

$$= \pi^{-2}\int d^2\alpha d^2\beta\langle\alpha|\rho|\beta\rangle|\alpha\rangle\langle\beta|$$

$$\langle\alpha|\rho|\beta\rangle = e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)}\sum_{n,m}\frac{\alpha^{*n}\beta^m}{\sqrt{n!}\sqrt{m!}}\langle n|\rho|m\rangle$$

$$\rho = \pi^{-2}\int d^2\alpha d^2\beta e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)}R\left(\alpha^*,\beta\right)|\alpha\rangle\langle\beta|$$

which is R – representation

where, $$R\left(\alpha^*,\beta\right) = \sum_{n,m}\frac{\alpha^{*n}\beta^m}{\sqrt{n!}\sqrt{m!}}\langle n|\rho|m\rangle$$

Characteristic functions:

Characteristics function is defined as

$$\chi(\xi) = Tr[\rho e^{\xi a^+ - \xi^* a}]$$

Normally ordered characteristic function is defined as,

$$\chi_N(\xi) = Tr[\rho e^{\xi a^+} e^{-\xi^* a}]$$

Normally Ordered coherence function is defined as

$$\Gamma_N^{(n,m)} = Tr[\rho a^{+n} a^m]$$

Relation between Characteristic function and Coherence function:

$$\frac{\partial}{\partial\xi}\left(e^{\xi a^+ - \xi^* a}\right) = a^+ e^{\xi a^+ - \xi^* a}$$

$$\frac{\partial}{\partial\xi^*}\left(e^{\xi a^+ - \xi^* a}\right) = e^{\xi a^+}\left(-ae^{-\xi^* a}\right)$$

$$\left(\frac{\partial}{\partial \xi}\right)^n \left(-\frac{\partial}{\partial \xi^*}\right)^m [e^{\xi a^+ - \xi^* a}]\Big|_{\xi=0} = a^{+n} e^{\xi a^+} a^m e^{-\xi^* a}\Big|_{\xi=0} = a^{+n} a^m$$

$$\Gamma_N^{(n,m)} = Tr[\rho a^{+n} a^m]$$

$$= Tr\left[\rho \left(\frac{\partial}{\partial \xi}\right)^n \left(-\frac{\partial}{\partial \xi^*}\right)^m [e^{\xi a^+ - \xi^* a}]\Big|_{\xi=0}\right]$$

$$= \left(\frac{\partial}{\partial \xi}\right)^n \left(-\frac{\partial}{\partial \xi^*}\right)^m Tr\left[\rho [e^{\xi a^+ - \xi^* a}]\Big|_{\xi=0}\right]$$

$$= (-1)^m \left(\frac{\partial}{\partial \xi}\right)^n \left(\frac{\partial}{\partial \xi^*}\right)^m \chi_N(\xi)\Big|_{\xi=0}$$

$$\Gamma_N^{(n,m)} = (-1)^m \left(\frac{\partial^{n+m}}{\partial \xi^n \partial \xi^{*m}}\right) \chi_N(\xi)\Big|_{\xi=0}$$

Properties:

1. $\chi_N(\xi) = Tr[\rho e^{\xi a^+ - \xi^* a}]$

$\chi_N(0) = Tr[\rho] = 1$

2. $\chi_N(\xi) = Tr[\rho e^{\xi a^+ - \xi^* a}]$

$$(\chi_N(\xi))^* = \left(Tr[\rho e^{\xi a^+ - \xi^* a}]\right)^*$$

$$= Tr[\rho e^{\xi a^+ - \xi^* a}]^*$$

$$= Tr[e^{-\xi a^+} e^{\xi^* a} \rho] \qquad \text{since } \rho \text{ is hermitian}$$

$$= Tr[\rho e^{-\xi a^+} e^{\xi^* a}] \qquad \text{since } \rho \text{ is hermitian}$$

$$= Tr[\rho e^{-\xi a^+ - (-\xi^* a)}]$$

$$= \chi_N(-\xi)$$

Antinormally Ordered Characteristic function:

Antinormally ordered characteristic function is defined as,

$$\chi_A(\xi) = Tr[\rho e^{-\xi^* a} e^{\xi a^+}]$$

Antinormally Ordered coherence function is defined as

$$\Gamma_A^{(n,m)} = Tr[\rho a^n a^{+m}]$$

Hence

$$\Gamma_A^{(n,m)} = (-1)^m \left(\frac{\partial^{n+m}}{\partial \xi^n \partial \xi^{*m}} \right) \chi_A(\xi)\Big|_{\xi=0}$$

$$\chi_A(0) = Tr[\rho] = 1$$

$$(\chi_A(\xi))^* = \chi_A(-\xi)$$

Interrelationship between characteristics function:

BCH identity is

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \quad \text{or}$$

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$$

$$e^{A+B} = e^B e^A e^{\frac{1}{2}[A,B]}$$

$$A = \xi a^+, \; B = -\xi^* a$$

$$[A, B = [\xi a^+, \xi^* a] = |\xi|^2$$

$$e^{\xi a^+ - \xi^* a}] = e^{-\frac{1}{2}|\xi|^2} e^{\xi a^+} e^{-\xi^* a}$$

$$\text{Also } e^{\xi a^+ - \xi^* a}] = e^{\frac{1}{2}|\xi|^2} e^{-\xi^* a} e^{\xi a^+}$$

Characteristics function: $\chi_S(\xi) = \chi(\xi) = Tr[\rho e^{\xi a^+ - \xi^* a}]$

Normally Ordered Characteristics function: $\chi_N(\xi) = Tr[\rho e^{\xi a^+} e^{-\xi^* a}]$

Antinormally Ordered Characteristics function: $\chi_A(\xi) = Tr[\rho e^{-\xi^* a} e^{\xi a^+}]$

therefore,

$$\chi_S(\xi) = e^{-\frac{1}{2}|\xi|^2} \chi_N(\xi)$$

$$\chi_S(\xi) = e^{\frac{1}{2}|\xi|^2} \chi_A(\xi)$$

Problems:

1. Find the value of $\chi_N(\xi)$ and $\chi_A(\xi)$ for $\chi(\xi) = e^{-C|\xi|^2}$, where C is constant.
2. Evaluate $\Gamma_N^{(n,n)} = \mathrm{Tr}[\rho a^{+n} a^n]$ and $\Gamma_A^{(n,n)} = \mathrm{Tr}[\rho a^n a^{+n}]$.
3. Show that $\chi_N(\xi)$ and $P(\alpha)$ are Fourier transform to each other.
4. Show that $\chi_A(\xi)$ is Fourier transform of $\langle \alpha|\rho|\alpha \rangle$.